Applications of the electromagnetic field theory for calculations of the radiative thermal transfer in coaxial multi-layer systems

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Abstract In the present article we intend to accomplish a rigorous analysis which allows to elaborate useful symbolic and numerical codes leading to an accurate evaluation of the thermal radiation. We will consider the case of the electromagnetic field in axially symmetric systems, a case of real interest in cryogenics.

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I. ELECTROMAGNETIC FIELD IN CONDUCTIVE BOUNDED CYLINDRICAL SYSTEMS

The first topic is the electromagnetic field inside a hollow cylinder cavity with conducting walls and the spectrum of resonance frequencies associated to the various modes of the field.

We will assume that the field has a monochromatic time dependence $e^{i\omega t}$. The symmetry axis will be taken as the z-axis of the coordinate system. With this choice, the field will split in two orthogonal components as

$$\vec{E} = \vec{e}_z E_z + \vec{E}_\perp \tag{1.1}$$

The walls of the cavity are situated at $\rho = b$, $z = 0$ and $z = l$.

The tangential component of the field should vanish on the conducting walls, that is $E_z = E_\phi = 0$ on the cylinder surface $\rho = b$ and $E_\rho = E_\phi = 0$ at $z = 0$and$z = l$ (fig.1). Inside the hollow cavity the field will obey the wave equation

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E} = 0$$

or

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\vec{E} = 0$$

for the case of harmonic field of frequency ω. The axial component of the space part of the electric field obeys the same equation:

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right)E_z(\rho, \phi, z) = 0 \tag{1.2}$$

and so does the transverse part $\vec{E}_\perp : (E_\rho, E_\phi)$. The equation can be written in cylindrical coordinates as

$$\left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} + \frac{\omega^2}{c^2}\right)E_z = 0 \tag{1.3}$$

and assumes a separate solution of the type

$$E_z(\rho, \phi, z) = R(\rho)\Phi(\phi)Z(z)$$

. Thus, the Helmholtz PDE above becomes

$$\frac{1}{R}\left(\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho}\right) + \frac{1}{\rho^2\Phi}\frac{d^2\Phi}{d\phi^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} + \frac{\omega^2}{c^2} = 0$$

or

$$\frac{1}{R}\left(\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho}\right) + \frac{1}{\rho^2\Phi}\frac{d^2\Phi}{d\phi^2} + \frac{\omega^2}{c^2} = -\frac{1}{Z}\frac{d^2Z}{dz^2} = k^2 \tag{1.4}$$

The reflections at the top $z = l$and bottom $z = 0$ surfaces impose a z-dependence appropriate for standing waves. Thus, in the eq. (1.6) the constant $k^2 \geq 0$ and the solution $Z(z)$ will be a combination of $\sin kz$and $\cos kz$ which

will obey the boundary conditions. Further, in order to have a uniform ϕ dependence, $\Phi(\phi)$ should be of the form $\Phi(\phi) = e^{\pm im\phi}$ with integer $m = 0, 1, 2, 3, ...$

Thus, the radial equation becomes

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(\gamma^2 - \frac{m^2}{\rho^2}\right) R = 0 \tag{1.5}$$

where $\gamma^2 = \frac{\omega^2}{c^2} - k^2$. The equation (1.7) is the Bessel equation of order $m =$ integer and the solution is a combination of Bessel and Neumann functions of order m , $J_m(\gamma\rho)$and$N_m(\gamma\rho)$. If we take into account that the solution should be finite for any ρ, including $\rho = 0$ and remember the behavior of Bessel and Neumann functions in the vicinity of the origin

$$J_m(x) \rightarrow \frac{1}{\Gamma(m+1)}\left(\frac{x}{2}\right)^m$$

respectively

$$N_m(x) \rightarrow -\frac{\Gamma(m)}{\pi}\left(\frac{2}{x}\right)^m$$

for $m \geq 1$ and

$$N_0(x) \rightarrow \frac{2}{\pi}\left[\ln\left(\frac{x}{2}\right) + 0.5772...\right]$$

than only $J_m(\gamma\rho)$ is acceptable, because the second solution is singular at the origin. In the case of Transverse Magnetic (TM) modes $H_z = 0$ everywhere in the cavity, while the transverse components of the electric and magnetic field are

$$\vec{E}_\perp = \frac{1}{\gamma}\nabla_\perp \frac{\partial E_z}{\partial z} \tag{1.6}$$

and, respectively

$$\vec{B}_\perp = \frac{i\omega}{c^2\gamma^2}\vec{e}_z \times \nabla_\perp E_z \tag{1.7}$$

where $\nabla_\perp : \left(\frac{\partial}{\partial\rho}, \frac{1}{\rho}\frac{\partial}{\partial\phi}\right)$ is the Hamilton's operator in polar coordinates.

Than E_z must assume the form

$$E_z(\rho, \phi, z) = E^0 J_m(\gamma\rho)e^{\pm im\phi}\cos\left(p\pi\frac{z}{l}\right) \text{with} p = 0, 1, 2, ... \tag{1.8}$$

in order that $\vec{E}_\perp$ to satisfy the boundary conditions at $z = 0$and$z = l$. Also, the boundary condition $E_z = 0$ for $\rho = b$ requires that the argument γb should be one of the zeros of the Bessel function J_m. Thus, γ will assume only a set of discrete values

$$\gamma_{mn} = \frac{x_{mn}}{b}$$

where x_{mn} is the n-th zero of J_m.

The general solution of the field equations (1.10) inside the cavity that obeys the appropriate boundary conditions will be a superposition of the normal modes :

$$E_z(\rho, \phi, z) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sum_{p=0}^{\infty} E_{mnp}J_m(x_{mn}\frac{\rho}{b})e^{\pm im\phi}\cos\left(p\pi\frac{z}{l}\right) \tag{1.9}$$

The resonance frequencies of the cavity will depend on three parameters and will be given in the case of TM modes by

$$\omega_{mnp} = c\sqrt{\left(\frac{x_{mn}}{b}\right)^2 + \left(\frac{p\pi}{l}\right)^2} \tag{1.10}$$

with m=0,1,2,... n=1,2,... p=0,1,2,...

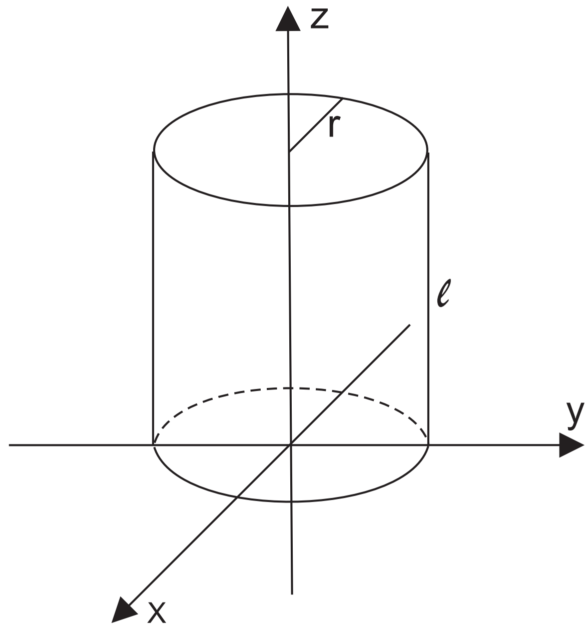

FIG. 1: The geometric parameters of the system

II. HOLLOW CYLINDRIC RING WITH CONDUCTING WALLS

The second topic is the electromagnetic field inside a hollow cavity of the form of a cylindric ring with conducting walls, with inner radius a, outer radius $b$and height l.

The PDE and the boundary conditions that the fields obey will be the same as in the case of the cylinder cavity, except that a supplementary boundary at $\rho = a(> 0)$ appears. Since the origin $\rho = 0$ no longer belongs to the domain of values that ρ assumes, than the particular solution of the radial equation will be now a linear combination of both Bessel and Neumann functions:

$$R_m(\rho) = A_m J_m(\gamma\rho) + B_m N_m(\gamma\rho) \tag{2.1}$$

In order that the field to satisfy homogenous boundary conditions on both $\rho = a$and $\rho = b$ boundaries, the radial funtion above should assume the form

$$R(\rho) = A\left[J_m(\gamma_{mn}\rho) - \frac{J_m(\gamma_{mn}a)}{N_m(\gamma_{mn}a)}N_m(\gamma_{mn}\rho)\right] \tag{2.2}$$

where γ_{mn},$n = 1, 2, 3, ...$ are the solutions of the equation

$$J_m(\gamma_{mn}b)N_m(\gamma_{mn}a) - J_m(\gamma_{mn}a)N_m(\gamma_{mn}b) = 0$$

They will be determined by appropriate numerical methods.

In the case of TM modes, the general solution for the axial component E_z is the linear combination:

$$E_z(\rho, \phi, z) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sum_{p=0}^{\infty} E_{mnp}\left[J_m(\gamma_{mn}\rho) - \frac{J_m(\gamma_{mn}a)}{N_m(\gamma_{mn}a)}N_m(\gamma_{mn}\rho)\right] e^{\pm im\phi}\cos\left(p\pi\frac{z}{l}\right) \tag{2.3}$$

while the resonant frequencies are

$$\omega_{mnp} = c\sqrt{(\gamma_{mn})^2 + \left(\frac{p\pi}{l}\right)^2} \tag{2.4}$$

with m=0,1,2,... n=1,2,... p=0,1,2,...

III. CONCLUSIONS

The presented formulae allow an exact calculus of the electric field inside axial cavities with conducting walls, which may be used for a rigorous evaluation of the radiation thermal flux in such a system. For cryogenic systems, the numerical calculations need such accurate formulae, taking into account that a large number of coaxial insulators will be used and any errors will propagate and will be recursively amplified. The only way to minimize the global errors is to keep them as low as possible at any stage, thus involving from the beginning these exact analytical solutions. That is why the numerical calculus of the special functions involved must be carefully conducted, implying an increased computing effort in order to precisely describe the total radiation heat transfer rate in cryogenic installations.

APPENDIX

The Bessel functions $J_\nu(x)$are solutions of Bessel equation

$$\left[\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} + \left(1 - \frac{\nu^2}{x^2}\right)\right] R(x) = 0 \tag{A.1}$$

and assume the following series expansions

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\Gamma(j+\nu+1)} \left(\frac{x}{2}\right)^{2j} \tag{A.2}$$

$$J_{-\nu}(x) = \left(\frac{x}{2}\right)^{-\nu} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\Gamma(j-\nu+1)} \left(\frac{x}{2}\right)^{2j}$$

They satisfy the following orthogonality relation on the interval $\rho \in [0, a]$

$$\int_0^a \rho J_\nu\left(x_{\nu n}\frac{\rho}{a}\right) J_\nu\left(x_{\nu k}\frac{\rho}{a}\right) d\rho = \frac{a^2}{2}\left[J_{\nu+1}\left(x_{\nu n}\right)\right]^2 \delta_{nk} \tag{A.3}$$

The above solutions are independent for all ν except the case of integer $\nu = m = 0, 1, 2...$ when

$$J_{-m}(x) = (-1)^m J_m(x)$$

In this case a second independent solution of Bessel equation is the Neumann function

$$N_\nu(x) = \frac{J_\nu(x)\cos\nu\pi - J_{-\nu}(x)}{\sin\nu\pi} \tag{A.4}$$

Both Bessel and Neumann functions as well as their combinations

$$H_\nu^{(1)}(x) = J_\nu(x) + iN_\nu(x)$$

and

$$H_\nu^{(2)}(x) = J_\nu(x) - iN_\nu(x)$$

i.e the Hankel functions, satisfy the recursion formulae

$$X_{\nu-1}(x) + X_{\nu+1}(x) = \frac{2\nu}{x} X_\nu(x) \tag{A.5}$$

and

$$X_{\nu-1}(x) - X_{\nu+1}(x) = 2\frac{dX_\nu(x)}{dx} \tag{A.6}$$

where X(x) denotes a Bessel, Neumann or Hankel function.

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